Dealer Quotes, Order Flow and Indirect Foreign Currency Utility in a Multiple Dealership Market

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DEALER QUOTES, ORDER FLOW AND INDIRECT FOREIGN CURRENCY UTILITY IN A MULTIPLE DEALERSHIP MARKET

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ABSTRACT

The paper proposes a model of multiple dealer forex trade in two variants: for direct and brokered market organization. The equilibrium order flow pattern is derived as a function of shadow prices (marginal valuations) of FX holdings across market participants. The shadow currency values can be heterogeneous due to differences in preferences, endowments or asset payoff information, giving rise to non-zero trades in equilibrium. Purchases (sales) of the currency are initiated by those market users whose marginal currency values are high (low) in relation to the marginal valuations of the market maker. As a consequence, one would observe a strong price impact of the order flow when it moves the marginal value of the recipient dealer away from the no-trade level, and a weak order flow impact when the received position reverts the marginal value back to it. The Nash equilibrium of the studied one period inter-dealer game is obtained as a steady state Nash equilibrium of a differential game between the same players. We also find that differences in equilibrium quoting behavior result in higher volumes of inter-dealer trade in the brokered market compared to the direct market. However, under both trading mechanisms hot potato trades are non-zero even if the dealers are perfectly symmetric. For a given level of non-dealer investor foreign currency demand, gains from trade in the brokered market are lower than in the direct market.

Keywords: forex microstructure, order flow, multiple dealership, trading mechanism
JEL Classification: F31; G15; C72

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1. Introduction

Current literature on FX microstructure identifies the client order flow as the prime source of new exchange rate information that is impounded in prices set by the dealers (Evans and Lyons, 2002). At the same time, the basic Evans-Lyons paradigm of exchange rate formation implies zero *net* client order flow, if aggregated across all market makers. This happens because the optimizing price setting by the dealers dictates the shift of the quoted price to the level allowing them to unwind undesired positions. Therefore, their equilibrium quotes induce the clients to generate "second stage" orders that would offset that part of their "first stage" orders made at the original quotes, which is in excess of the dealer-acceptable value. So, if the net client order flow may erroneously evoke an impression of no relation to the price change, what is the right order flow to look at?

One choice is the inter-dealer order flow, since it is generated by the dealers who redistribute the initially taken position risk coming from client trades. However, inter-dealer order flow would only reflect fundamental price information if the latter were dispersed asymmetrically among clients of individual dealers. On the contrary, there would not be any information revelation if all dealers received the same client order flow signal. Therefore, one of the directions, in which the basic Evans-Lyons paradigm can be extended, is the introduction of investors who choose between, and trade with several dealers simultaneously. Quite often, such traders are the most important carriers of fundamental price signals.

On the empirical side, the inter-dealer order flow is usually difficult to separate from the one coming from clients. Orders of both categories arrive in random sequence and, moreover, a dealer may be a carrier of even stronger exchange rate information than any of the non-dealer investors, so that his active trades are difficult to classify in accordance with the scheme outlined above. In other words, inter-dealer order flow includes, but is not limited to, "hot potato" trades of unwanted currency quantities. Therefore, primary and secondary reasons for inter-dealer order flow are impossible to distinguish in the data.

There is no single "price impact of the order flow" An alternative approach taken in this paper is based on the joint view of the active order of one party and the state of the other party. In the studied model, an order can have a big or a small price impact irrespective of whether it comes from a client or a dealer. Instead, the impact-determining factor is the marginal currency valuation by the order-receiving market maker. A low marginal valuation prompts the dealer to reduce position, high – to expand position and neutral - not to change position, i.e. not to use the market for own buys or sales. If the received order is big enough to cause a move from this neutral level to a low marginal valuation level (dictating a position reduction as the optimal action), or to a high level creating a preference for position expansion, then the order has a visible price impact. This is because the market maker would change his quotes in order to induce the corresponding subsequent trades with other market users. Conversely, if the received order results in a shift of the originally non-neutral marginal valuation towards the no-trade value, the price impact is negligible, since the market maker has no need to "invite" additional trades by shifting quotes. Altogether, we replace a one-dimensional picture of order flow impact on prices with a two-dimensional one, where the price change is a function of both incoming order and the current marginal valuation.

Another important feature of the model is the presence of investors who have access to all quotes and trade with all dealers at once, in quantities of their choice. That is, such an agent has the same possibility to use the inter-dealer market as the dealers themselves. The existence of these "global" investors is the reason why the observed order flow can serve as a source of the future price-relevant information for the dealer. This approach was earlier used in a decentralized dealer FX market model in continuous time studied in Derviz, 2003, 2004. Since the global investor orders are split among all market makers, whilst the order sign and volume are determined by her marginal valuation, privately received orders from global investors confer information about the overall direction in which the market is moving. This is much more than what can be read off the orders from "local" market users (exclusive clients of a given dealer). The latter only affect the dealer's endowment with which the subsequent inter-dealer trading round is entered with.

Gains from trade are higher in the direct than in the brokered inter-dealer forex To address the market organization issue, we construct two variants of the model, one for the direct and the other for brokered trading mechanism. In the brokered market, the dealer-client role difference only concerns the limit order submission (in the form of pricing schedules), which is reserved to the dealers. Both dealers and non-dealers submit market orders. Dealer pricing schedules are automatically crossed by the broker and then the integral "standing pricing schedule" is announced. This means that the dealer cannot fully determine the market user order by means of his own pricing schedule, even if he takes the market user demand as given. The non-dealer investor in the brokered market is only given access to the standing pricing schedule. Her order is split proportionally between dealers, based on the distribution of their quoting parameters. Therefore, the market user cannot split orders herself. Altogether, the presumed single price advantage of the brokered market has a cost: we show that the ability of investors to adjust their marginal foreign cash utility by trade is limited compared to the direct market.

In the direct market, a market user's optimal behavior consists in splitting orders between all market makers in proportions that equalize marginal values of purchased/sold quantities across contacted dealers. This property of our model is similar to that of Bernhardt and Hughson, 1997, and Menkveld, 2001.

Price signals of inter-dealer trades cannot be reduced to those of the clients The model possesses a Nash equilibrium that equalizes the market users' indirect marginal utilities of foreign cash and the effective quotes of the market makers. The properties of this equilibrium show that the order flows (of both dealers in the inter-dealer market and their customers) are functions of these marginal indirect utilities, and so are the quoting parameters (i.e. the pricing schedule slope and intercept) of the dealers. Although we do not study imperfect or incomplete information game settings, the chosen benchmark concept of the Nash equilibrium in the normal form inter-dealer game still broadens our understanding of learning from order flow by the dealers. The learning duration is compressed to one period and is, therefore, more implicit than is usual in the theoretical microstructure literature. Nevertheless, it allows us to model equilibrium outcomes in the inter-dealer market in situations when one cannot make a reasonable assumption about the "true" statistical law of uncertainty. That is, the dealers in our model do not make inferences about the stationary distribution of a fundamental factor behind the currency value, but go directly to inferring the instantaneous internal valuation of the currency itself by the counter-parties. The equilibrium is obtained by embedding the original static game into a differential game with the same players and deriving its steady state Nash equilibrium conditions by means of the maximum principle.

In the static game setting, we generate dealer heterogeneity by varying their exogenous endowments of domestic and foreign cash. Variation in the same parameter of the non-dealer investor is the source of the customer order flow. However, the present model does not belong to the inventory trade category, since equilibrium trade patterns are determined by investor *marginal valuations* and not inventory levels directly. In the underlying dynamic game (see Section 4), the marginal currency value is a sufficient statistic that summarizes the impact of beliefs, asset returns and institutional status, on the currency demand or supply. Trades serve to discover the marginal currency values of the counter-parties. This property manifests itself in many ways, among them the ability of the model to generate non-trivial hot potato trades in equilibrium, even when dealer inventories are identical. The dealer directs orders toward other dealers even when he knows that the very same order will be directed back to him. This is so because directing an order towards another market maker to rebalance one's position dominates the cooperative no-trade strategy in the one-shot game. Potential collusion in a repeated game is not studied, since we believe it to be of little practical importance in the market where many dealers compete for global client orders.

Structure of the paper: Section 2 defines the model, the agents' optimization problems and optimal active trade rules at given quotes (pricing schedules). This is first done for the global non-dealer investor, and then for the investor with a market-maker function (dealer) in both direct inter-dealer and brokered inter-dealer market settings. In Section 3, we derive the optimal quoting conditions in the direct and brokered markets and discuss consequences concerning the marginal currency value role in the price impact of the order flow. Section 4 outlines an imbedding of the one-shot inter-dealer game into a differential game. This embedding facilitates the Nash equilibrium calculation, described in the technical Appendix, together with several proposition proofs. The second subsection of Section 4 discusses the outcomes of numerical NE calculations for the one-period game and the corresponding "reduced form" relations between order flows, prices and marginal FX values. Section 5 concludes.

1. The Model

In all variants of the model, we shall consider a dealer, i.e. an agent who makes the market by posting quotes, and an investor, who trades at dealers' quotes without placing her own. However, both types of agents will act as *market users* when they place their own orders. State variables and preferences of all agents will have the same structure. It is assumed for definiteness that all agents are domestic residents, but this assumption is not pivotal for the results.

Let x^m be the domestic cash holdings and x^i – foreign cash holdings of a given investor. There is an exogenous endowment y^m of domestic cash and y^i – of foreign cash. The liquid wealth of the market participant is valued by means of the utility function $(x^m, x^i) \mapsto v(x^m,x^i)$. This function is strictly increasing and strictly concave in each argument and converges to $-\infty$ when either argument goes to $-\infty$. An example is

$$v(x^m, x^i) = g - b_m e^{-a_m x^m} - b_i e^{-a_i x^i}, \tag{1}$$

with positive constants a_m, a_i, b_m, b_i and g. This definition of preferences implies that the agent's domestic and foreign assets (including, but not limited to, cash) are imperfect substitutes. The agent may have short selling constraints, limits on open positions or other reasons to avoid imbalances in the currency composition of his/her portfolio. Thus, negative cash holdings in either currency (debt, short position) are penalized whereas positive holdings have decreasing marginal value.

Each dealer quotes a pricing schedule that is a smooth convex transform of a linear schedule of the form

$$p=r+sq,$$

where p is the transformed price, q is the quantity bid/offered ($q<0$ corresponds to purchases from the client and $q>0$ – to sales to the client) and r, s are constants, $s>0$. The price is obtained by the rule $P=f(p)$, where f is a strictly positive, increasing and convex function on the real line. Our standard example will be $f(p)=e^{cp}$, for some positive constant c. The fact that we do not use linear pricing schedules directly, as e.g. in Kyle, 1985, and many papers following Kyle's, is explained by our effort to obtain internal solutions for individual investor problems and later – a well defined Nash equilibrium without the need to check whether the resulting prices are positive. This is particularly important in the settings where a closed form solution cannot be derived and numerical methods are used to calculate equilibrium transaction patterns.

In order to economize on notation and highlight the most important qualitative results, we shall concentrate attention on the case when there are two dealers in the FX market (indexed by 1 and 2). There is also a single representative *global* non-dealer investor, indexed by U. The word "global" here means that this market user approaches both dealers for quotes and trades.

1.1 Decentralized dealership

Trades in the market considered in this subsection are fully non-transparent: only bilateral quoting and trade between dealers and other investors exists. Since every agent obtains both dealer quotes, intransparency refers to the *volume* of effectuated trades other than one's own, i.e. the former are unobservable. Therefore, transaction prices are also unobservable globally until the end of the period. We do not assign informational significance to any price signal, since this version of the model does not consider stationary distributions of uncertainty factors. (Were such distributions a part of the considered equilibrium of the game, they would be inferred by the dealers jointly from the orders and prices in the course of Bayesian learning. However, in most real life FX markets, it is doubtful whether the assumption of stationary risk factors driving market user demands, is justified.)

2.1.1 Global non-dealer market user

We denote by r^i, s^i the parameters of the pricing schedule quoted by dealer j, j=1,2. The orders placed by the investor at these quotes with the two dealers are denoted by Q^j, j=1,2 ($Q^j>0$ if the order is for the foreign cash purchase). Then the end-of-period domestic and foreign cash holdings of the investor are:

$$x^m = y^m - f(r^1 + s^1 Q^1)Q^1 - f(r^2 + s^2 Q^2)Q^2, \tag{2a}$$

$$x^i = y^i + Q^1 + Q^2. \tag{2b}$$

The investor maximizes (1) subject to (2). The parameters (r^j, s^j), j=1,2, of the market-maker quoting schedules are taken as given by the market user.

Let x_m, x_i be the investor's *marginal utilities* of the domestic and foreign cash, i.e.

$$x_m = \frac{\partial v}{\partial x^m}(x^m, x^i) = v_m(x),\; x_i = \frac{\partial v}{\partial x^i}(x^m, x^i) = v_i(x).$$

We shall also call these partial derivatives the *investor's shadow prices* of the domestic and foreign currency, respectively.

For us, the most important part of optimal policies is the one which describes the currency purchases/sales $Q^{1,2}$ from/to both dealers. Let us denote by

$$q = \frac{x_i}{x_m} = \frac{v_i(x)}{v_m(x)} = \frac{a_i b_i}{a_m b_m} e^{a_m x^m - a_i x^i}$$

the relative *marginal valuation* of foreign currency by the market user. Then the optimal trades can be written in the form

$$Q^j = \frac{g\left(r^j, q\right)}{s^j}, \; j=1,2, \tag{3}$$

where g is the implicit function solving the equation

$$f'\left(r^j + g(r^j, q)\right) g(r^j, q) + f\left(r^j + g(r^j, q)\right) = q \tag{4}$$

identically for all r^j and q. Such a solution exists if $0 < 2f' + f''g$ in the range of considered parameters r^j and q. Then g, which is the optimal traded quantity per unit of pricing schedule slope s, is a smooth function with partial derivatives (in the future, we denote them by subscripts) given by

$$g_r = -\frac{f' + f''g}{2f' + f''g},\; g_q = \frac{1}{2f' + f''g}. \tag{5}$$

Expressions (5) show that optimal currency demand/supply "per unit of spread", denoted by g, is decreasing in the mid-quote and increasing in the marginal foreign currency valuation. These are intuitively correct properties of optimal orders: the market user buys more (sells less) of the foreign currency when the pricing schedule intercept r that she faces, goes down and when her marginal valuation of the foreign currency goes up. The marginal valuation parameter q itself has an interpretation of the market user demand intercept.

For our specific example of quoting rule $f(p)=e^{cp}$, the condition for the existence of internal solution for optimal orders is given by $g>-2/c$, equivalently, $Q>-2/(cs)$, and the partial derivative of g w.r.t. r is equal to

$$g_r = -\frac{1+cg}{2+cg} = -\frac{1+csQ}{2+csQ}.$$

2.1.2 Dealer's active trades

Formally, the decision problem faced by the dealer is an extension of the one solved by the non-dealer investor. For definiteness, the exposition in this subsection concerns dealer 1, with the formulations for dealer 2 being obtained by substitution of indices.

At the start of the trading period, dealer 1 quotes a pricing schedule (r^1,s^1), which is good for both the other dealer and the non-dealer global market user. Symmetrically, dealer 2 does the same, so that his pricing schedule (r^2,s^2) is good for dealer 1 to trade at. Commonality of the dealer's quotes given to all market users, even in the decentralized market, is an important feature allowing us to simplify the price impulse propagation modeling. We justify it by referring to the reputational considerations on the part of the dealer.

As is common in strategic trade models, some sort of noise- (i.e. not fully rational) traders are needed to generate non-trivial transactions in the market. Here, noise traders will be introduced to ensure non-zero exogenous quoting costs of the dealers. Specifically, dealers will always face a situation in which the optimal choice of the pricing schedule slope s will be strictly positive and finite. Any attempt to increase utility by reducing s to zero or infinitely increasing it will result in an increasingly costly position received from a noise trader.

Formally, we shall assume that, after active trades of dealers 1 and 2 and market user U have been decided upon, the pricing schedule of each market maker can be randomly matched directly with a similar schedule of an outside trader (i.e. someone whom we do not model explicitly). In this way, a specific additional trade is generated at the dealer's quote. This external matching can be a consequence of the dealer's bargaining with an important local customer (i.e. someone trading only with this dealer, as opposed to the global investor whom we model explicitly), automatic crossing by a voice broker or another unspecified reason. (In the part dedicated to the brokered FX market, a similar automatic schedule crossing between contributors to the order book will be an essential part of the brokerage process that will lead to a common pricing schedule, or state of the book.) It is convenient to assume that the outside trader pricing schedule slope is equal to that of the dealer. Technically, this element of the model allows one to obtain an internal solution for the price schedule slope, without affecting the main qualitative properties of equilibrium.

Matching happens by adding up the pricing schedules horizontally and effectuating the transaction at the price that equalizes one dealer's demanded quantity with the other's supplied quantity. Specifically, let $q = \frac{p - r^j}{s^j}$ be the pricing schedule of dealer j and $q = \frac{p - g^j}{s^j}$ - the schedule of the local trader (we will work in the space of transformed prices p). Then their joint schedule is $q = \frac{p - r^{cj}}{s^{cj}}$, where $r^{cj} = \frac{r^j + g^j}{2}$, $s^{cj} = \frac{s^j}{2}$. By substituting the transformed price r^{cj} into the two dealers' pricing schedules, we see that at this price, dealer j is willing to transact quantity $q^{cj} = \frac{g^j - r^j}{2s^j}$ (sell if $g^j > r^j$, buy in the opposite case), whereas the outside trader is willing to transact minus this quantity, i.e. $\frac{r^j - g^j}{2s^j}$.

Matching of the price schedule happens at a cost, which we define in domestic cash units. These costs can be associated with the required provisions that the dealer must make for the eventuality of noise trader matching. Specifically, let h be a strictly increasing and strictly concave function on the real line with $h(0)=0$, $h'(0)=1$. When, as a result of price schedule matching with an outside trader, the dealer sells (buys) q^c units of foreign currency, he receives (pays the negative of) $f(q^c)h(q^c)$ units of domestic cash, which is less (more in absolute value) than $f(q^c)q^c$. Our principal example of the *transaction function* will be the linear-quadratic function $h(q)=q-aq^2/2$, $a>0$ a constant. The origin of non-linear cost of matching is in the precautions the dealer must take in order to cope with situations when the transacted volume resulting from this trade gets out of his control. That is the reason for convex growth of costs with volume. As a consequence of non-linearly growing matching costs, the dealer usually prefers to set a strictly positive pricing schedule slope s. Otherwise, he might be exposed to suboptimally high transaction volumes with the outside traders.

Further, define by q^{12} and q^{21}, the volumes of regular active trades directed by dealer 1 towards dealer 2 and vice versa. We shall assume that dealer 1 has a certain extent of market power over both market users who trade with him at his quotes. Namely, in his optimization problem, dealer 1 takes into account the demand/supply schedule (3) of the market user U and a similar schedule of dealer 2 (we shall see shortly that each dealer's active trades, indeed, satisfy an analogue of (3)). We shall denote by q^2 the marginal currency valuation of dealer 2 and use superscript U for the market user variables. Dealer 1 has end of period cash positions

$$x^m = y^m + f\left(r^1 + g(r^1,q^2)\right)\frac{g(r^1,q^2)}{s^1} + f\left(r^1 + g(r^1,q^U)\right)\frac{g(r^1,q^U)}{s^1}$$
$$- f\left(r^2 + s^2 q^{12}\right)q^{12} + f(r^{c1})h(q^{c1}) - c, \tag{6a}$$

$$x^i = y^i - \frac{g(r^1,q^2) + g(r^1,q^U)}{s^1} + q^{12} - q^{c1}. \tag{6b}$$

The objective function of dealer 1 is defined by (1) as for any other market participant. It is now obvious that equations for optimal active trades are the same as (3), but with Q^2 replaced by q^{12}

and q^U – by q^1. Therefore, by symmetry, the assumptions made in (6) about the dependence of q^{21} on r^1 and s^1 are validated.

Discussion of optimal quoting by the dealers is postponed until Section 3.

2.2 Brokered market

The brokered inter-dealer FX market (real-life prototypes being either EBS or Reuters Dealing 2000-2 electronic brokerage systems) will be modeled as an institution to which both dealers submit their pricing schedules $q = \frac{p - r^j}{s^j}$, j=1,2, as defined earlier. These schedules are added up horizontally to generate the *standing market schedule* $q = \frac{p - r^b}{s^b}$ with $r^b = \frac{s^2 r^1 + s^1 r^2}{s^1 + s^2}$, $s^b = \frac{s^1 s^2}{s^1 + s^2}$. At this automatic pricing schedule crossing occasion, dealer 1 sells to (if $r^2 > r^1$, otherwise buys the negative of this from) dealer 2 the amount $q^{b1} = \frac{r^2 - r^1}{s^1 + s^2}$ at price $f(r^b)$. The order at which the automatic pairwise crossing is conducted by the broker, affects the involved transaction prices, but not the automatically determined overall transaction volume of a given dealer with the rest. So, if one intends to generalize the current set-up to the case of three or more dealers, it is easiest to think of a broker who adds up individual pricing schedules horizontally at one batch, effectuating the transactions at a singe clearing price.

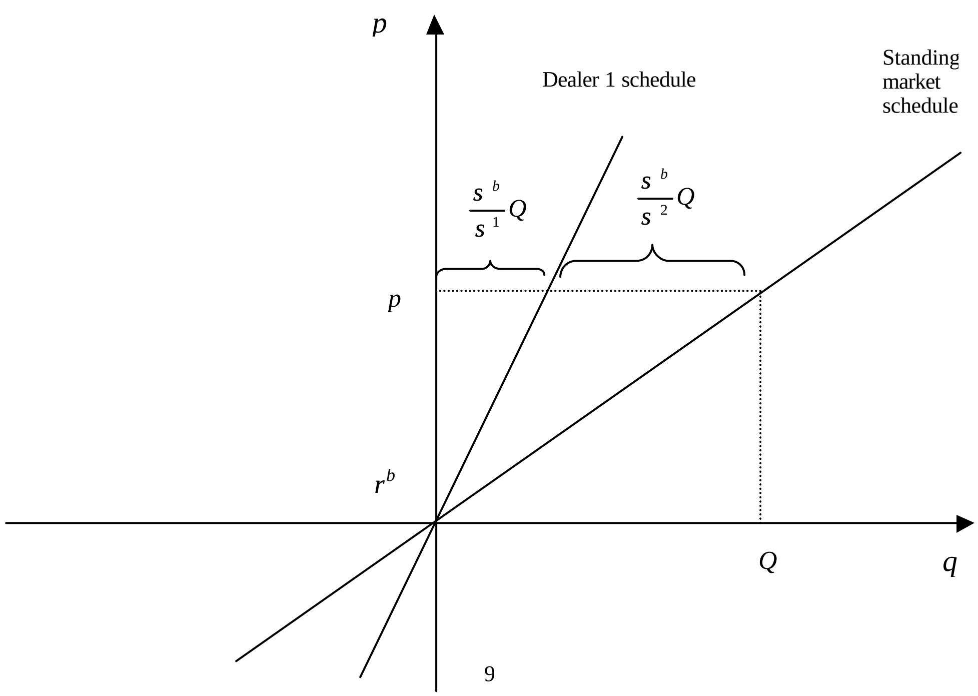

Fig. 1 State of the limit order book (the standing pricing schedule structure) and market order matching by the broker

Note: Q denotes the incoming market order (originating from either dealer or non-dealer investor) at the standing market schedule with parameters (r^b, s^b). This schedule is obtained by crossing the two dealer pricing schedules and adjusting to a common intercept r^b. The order is cleared at price $f(p)=f(r^b+s^bQ)$, whereas the order proportions are given by the relative slopes of individual dealer pricing schedules, equal to $\left(\frac{s^b}{s^1}, \frac{s^b}{s^2}\right)$ and adding up to unity.

In the same way as in the direct inter-dealer market case, we need to define external costs of quoting, caused by the existence of noise traders. We assume that each dealer can be randomly matched by the same noise trader as defined in 2.1.2 for the direct market, instead of being matched with the other dealer(s) by the broker. Prudential risk management rules of the dealer bank then require that the dealer creates a "corrective item" equal to $f(r^{c1})h(q^{c1})$ in x^m and the other corrective item $-q^{c1}$ in x^i to cover for this contingency. The presence of these terms restricts the domain of admissible pricing schedule slopes s^1 and guarantees the existence of an internal solution to the dealer optimization problem.

Upon completion of the automatic crossing procedure, the broker announces the standing pricing schedule to the two dealers and the non-dealer investor. This schedule is viewed as a horizontal sum of *reduced schedules of individual dealers*. This means that the original dealer schedule is reduced horizontally by the volume sold (or bought if $r^2<r^1$) in the course of automatic crossing. The resulting schedule is then $q=\frac{p-r^1}{s^1}-q^{b1}=\frac{p-r^b}{s^1}$ for dealer 1 and $q=\frac{p-r^2}{s^2}-q^{b2}=\frac{p-r^b}{s^2}$ for dealer 2, giving the standing schedule in sum.

Each of the three market users (two dealers and the non-dealer investor) then submit their orders to the broker who executes them at the standing schedule, by splitting each order Q at proportions $\left(\frac{s^b}{s^1}, \frac{s^b}{s^2}\right)$ among the two constituent schedules. Geometrically, this is equivalent to setting the order proportion of a dealer by letting his reduced schedule intersect the horizontal line, which goes through the standing schedule point with 1st coordinate Q (see Fig. 1).

Note that the brokered market set-up formulation requires a resolution of the following dilemma. Should the market order submitted by the dealer be partially returned to himself in accordance to the general rule defined above, or canceled out in some way by the dealer who is supposed to decide on market orders and quotes (limit orders) simultaneously? Fortunately, the formal outcome of the dealer optimization problem resolution does not depend on which variant is chosen. Here, we are taking the former view (a representative dealer partially "trading with himself"), for reasons of expositional transparency. One can justify this by assuming that active trades and quotes are decided upon by separate units within the same firm.

We proceed by deriving individual market participants' optimization problems.

2.2.1 Non-dealer global market user

At the beginning of the trading period, the non-dealer investor who uses the brokered market faces only the standing pricing schedule $P = f\left(r^{b} + s^{b}Q\right)$, where Q is the market-user's order size. She maximizes (1) with respect to Q, given the current domestic and foreign cash holdings. Her end-of-period cash holdings are

$$x^{m} = y^{m} - f(r^{b} + s^{b}Q)Q - c\,, \tag{7a}$$

$$x^{i} = y^{i} + Q\,. \tag{7b}$$

Therefore, the first order condition of optimality is

$$f'\left(r^{b} + s^{b}Q\right)s^{b}Q + f\left(r^{b} + s^{b}Q\right) = q^{U}\,,$$

similarly to what we have established in 2.1.1 for the direct market case, and the optimal order size is equal to

$$Q = \frac{g\left(r^{b}, q^{U}\right)}{s^{b}}\,. \tag{8}$$

2.2.2 Dealer's problem

Dealer 1 who uses the brokered market decides upon the same variable as the non-dealer investor (his order size, denoted by q^{1}), but also sets the parameters (r^{1}, s^{1}) of his pricing schedule that would be incorporated in the standing pricing schedule by the broker. He knows and takes into account the functional form of the non-dealer order (8) and a similar order $q^{2} = \frac{g\left(r^{b}, q^{2}\right)}{s^{b}}$ which is placed with the broker by dealer 2 (we shall see in a moment that his own optimal order is consistent with this assumption). From the broker, dealer 1 receives an order of size q^{b1} at price $f(r^{b})$ resulting from automatic price schedule crossing and proportion $\frac{s^{b}}{s^{1}}$ of orders Q and q^{2} (see above). Altogether, his end of period cash holdings are

$$x^{m} = y^{m} + \frac{f\left(r^{b} + g(r^{b}, q^{1})\right)g(r^{b}, q^{1})}{s^{1}}$$

$$+\frac{s^b}{s^1}\left\{ f\left(r^b+g(r^b,q^2)\right)\frac{g(r^b,q^2)}{s^b}+f\left(r^b+g(r^b,q^U)\right)\frac{g(r^b,q^U)}{s^b}\right\}$$

$$-f\left(r^b+s^bq^1\right)q^1+f(r^b)q^{b1}+f(r^{c1})h(q^{c1})-c, \tag{9a}$$

$$x^i=y^i-\frac{s^b}{s^1}\frac{g(r^b,q^1)+g(r^b,q^2)+g(r^b,q^2)}{s^b}+q^1-q^{b1}-q^{c1}. \tag{9b}$$

Observe the double appearance of the dealer 1 order $q^1=\frac{g\left(r^b,q^1\right)}{s^b}$ in the above equations. As mentioned earlier, this order returns to the dealer as the proportion $\frac{s^b}{s^1}$ of the current market order and at the same time is processed as his own market order. We shall see immediately that this understanding is internally consistent.

The first order condition of optimality for the dealer's market order is

$$f'\left(r^b+s^bq^1\right)s^bq^1+f\left(r^b+s^bq^1\right)=q^1.$$

Therefore, $q^1=\frac{g\left(r^b,q^1\right)}{s^b}$, $q^2=\frac{g\left(r^b,q^2\right)}{s^b}$, which is consistent with the assumption (regarding dealer 2 order size) used in (9).

3 Dealer's optimal quoting policy

We return to the dealer 1 optimization problem from Section 2 to characterize the optimal quoting strategies (i.e. the choice of parameters r and s). As before, we discuss the decisions of dealer 1 so that the other one's optimal moves are derived by symmetry.

Let us introduce the following notations:

$$a(r,q)=1+g_r(r,q)=\frac{f'\left(r+g(r,q)\right)}{2f'\left(r+g(r,q)\right)+g(r,q)f''\left(r+g(r,q)\right)},$$

$$P(r,q)=f\left(r+g(r,q)\right),\ P^b=f(r^b),\ P^c=f(r^c).$$

$P(r^1,q^2)$ ($P(r^1,q^U)$, P^b, P^{c1}) is the effective transaction price at which the order of dealer 2 (non-dealer market user, dealer 2 in the course of the price schedule crossing by the broker, noise trader by price schedule crossing) is executed by dealer 1 in the direct market (or brokered market in the case of P^b). We consider prices $P(r^1,q^2)$ and $P(r^1,q^U)$, as well as auxiliary parameters a as functions of quoting parameter (transformed mid-quote price) r^1 and the marginal valuation parameters q^2 and q^U, respectively. Similar understanding is employed in the brokered market context. Since the mid-quote transformed price r^b is a function of all four quoting parameters, the same is true for P^b. On the other hand, r^{c1} and P^c only depend on r^1, s^1.

Further, let us define functions

$$C^1(r,s)=\left[1+\frac{a(r-g^1)}{2s}\right]f\left(\frac{r+g^1}{2}\right)-q^1,\ D^1(r,s)=\frac{r-g^1}{2}\left[1+\frac{a(r-g^1)}{4s}\right]f'\left(\frac{r+g^1}{2}\right),$$

$$L^{p1}(r,q)=a\left(r,q^2\right)q^2+a\left(r,q^U\right)q^U+\left(2-a\left(r,q^2\right)-a\left(r,q^U\right)\right)q^1)-P(r,q^2)-P(r,q^U),$$

$$L^{s1}(r,q)=g(r,q^2)P\left(r,q^2\right)+g(r,q^U)P\left(r,q^U\right)-\left[g(r,q^2)+g(r,q^U)\right]q^1.$$

Analogous variables (index 2 everywhere replacing 1, and vice versa) are defined for dealer 2.

Direct inter-dealer market

Proposition 1 *The first-order conditions for the objective function optimization w.r.t. quoting parameters r^1 and s^1 of dealer 1 are given by equations*

$$L^{p1}(r^1;q^1,q^2,q^U)=\frac{1}{2}\left[C^1(r^1,s^1)+D^1(r^1,s^1)\right], \tag{10a}$$

$$L^{s1}(r^1;q^1,q^2,q^U)=\frac{r^1-g^1}{2}C^1(r^1,s^1). \tag{10b}$$

Calculations leading to (10) are given in the Appendix, Section A1.

An analogous pair of equations is valid for optimization of r^2 and s^2 by dealer 2.

Brokered inter-dealer market

We will need additional notation, which we introduce by putting

$$B^1(r^1,r^2,s^1,s^2)=\frac{s^1}{s^2}\left[f(r^b)-P(r^b,q^1)\right]+\frac{s^1(r^1-r^2)}{s^1+s^2}f'(r^b).$$

Proposition 2 *The first-order conditions for the current value Hamiltonian optimization w.r.t. quoting parameters r^1 and s^1 of dealer 1 are given by equations*

$$B^1(r^1, r^2, s^1, s^2) + \frac{s^1 + s^2}{2s^2}\left[C^1(r^1, s^1) + D^1(r^1, s^1)\right] = L^{p1}(r^b; q^1, q^2, q^U)\,, \tag{11a}$$

$$\frac{r^1 - g^1}{2} C^1(r^1, s^1) - \frac{s^1 s^2 (r^1 - r^2)}{\left(s^1 + s^2\right)^2}\left[L^{p1}(r^b; q^1, q^2, q^U) - B^1(r^1, r^2, s^1, s^2)\right]$$

$$= L^{s1}(r^b; q^1, q^2, q^U)\,. \tag{11b}$$

See Section A1 of the Appendix for the proof.

We now define a normal form game between dealer 1 and dealer 2, with the action space of each player consisting of his quoting parameters r and s, $s>0$, and his payoff given by (1) subject to (6) (in the direct market) or (9) (in the brokered market). Optimal active trade rules of the dealers, same as the non-dealer investor, stated in (3) or (8), are taken as given. Investors' cash endowments y^m and y^i are exogenous. For given marginal currency valuation parameters q^1, q^2 and q^U, the four equations (10) for the direct inter-dealer market and (11) for the brokered market, would give a pair of quoting rules. (The latter themselves determine the end-of-period cash positions of the players and, through them and the marginal utilities, also the marginal valuations q^1, q^2, q^U.) By fixing the parameters of the non-dealer investor and the other dealer, one can regard the optimal quoting rule (r, s) of each dealer that solves (10) (or (11)) as a *reaction function*. Their intersection gives the Nash equilibrium of the one-period trading game between dealers 1 and 2. Accordingly, the Nash equilibrium is the 10-dimensional vector (x, r, s) of the investor cash holdings and the dealer quoting parameters that satisfy the ten equations (2), (6), (10) for the direct market, and (7), (9), (11) for the brokered market.

Before starting to calculate this equilibrium, we can observe that (10b) and (11b) imply the following expression for the marginal currency valuation of the dealer, in the case when the noise trades are negligible:

$$q^1 \approx \frac{q^{21}}{q^{21} + Q^1} P(r^1, q^2) + \frac{Q^1}{q^{21} + Q^1} P(r^1, q^U)\,,\; q^1 \approx \frac{q^2}{q^2 + Q} P(r^b, q^2) + \frac{Q}{q^2 + Q} P(r^b, q^U)\,. \tag{12}$$

A straightforward analogue of this equation happens is valid for generalizations of the model to the case of more than two dealers. Qualitatively, (12) and its generalizations mean that the dealer's marginal currency value in any one-shot equilibrium is equal to *the sum of effective transaction prices paid by the remaining market participants weighted by their normalized orders*. This opens the way to empirical estimates of the dealer marginal values.

4 Equilibrium of the static inter-dealer game and the steady state equilibrium of a differential game

This section proceeds with first, explaining the method of equilibrium derivation in the inter-dealer game and next, discussing the properties of equilibrium prices and trades obtained numerically. The possibility of numerical solution is a direct consequence of the relation between the original static game and the more general differential game.

4.1 Embedding of the one period game

Calculation of the inter-dealer game Nash equilibrium as defined in the previous section requires solving a system of non-linear equations for the choice variables of market participants. Such a solution cannot be obtained in closed form, and even its numerical calculation by standard methods proves to be computationally difficult. There exists a way around this difficulty, based on constructing an alternative procedure for equilibrium derivation. It consists in defining a dynamic (differential) game for the same set of players and, basically, the same rules as the one-shot inter-dealer game discussed before. The exogenous parameters of this differential game can then be chosen in such a way that any steady state equilibrium of the dynamic game is equivalent to the Nash equilibrium of the original static one. However, the dynamic perspective will help us prove a number of properties of the steady state equilibrium that will simplify its calculation. Thanks to these simplifications, we are able to obtain the numerical solution to the Nash equilibrium problem. The dynamic game is also of interest in its own right, since it broadens our understanding of possible forex trade patterns in continuous time. Unfortunately, the complete characterization of Nash equilibria is complicated by a highly involved stability structure of the corresponding dynamic system. Nevertheless, by finding out this complex structure in the dynamic model, we should be less surprised by the excess volatility observed in the real-life forex. We also see that there should be a link from the institutional arrangements prevailing in the market to the volatility magnitude.

Below, we give a brief outline of the differential game in which the one-period game of Section 3 can be embedded as a steady state.

Now we will deal with flow variables (trade orders q, Q) in the form of rates per infinitesimal time period dt. Also, we define an exogenous *endowment rate* y^m (endowment per period dt) of domestic cash and a similar endowment rate y^i of foreign cash. For simplicity, we only consider constant cash endowment rates in this paper.

A consumption/dividend payment rate c is subtracted from the current domestic cash holdings. The dividends are evaluated by the period utility function $c \mapsto u(c)$ with standard properties (increasing, strictly concave). With a certain infinitesimal probability, the investor may need to stop operations in the current period and submit his/her book to an audit, which evaluates the cash holdings by means of a "liquidation" value function $(x^m, x^i) \mapsto v(x^m, x^i)$, i.e. the same as the wealth utility function of the static problem of Sections 2, 3.

We shall assume that the arrival of the said liquidation event is a Poissonian random event with intensity b. With probability e^{-bdt}, the operation will be continued in the immediate infinitesimal time interval dt after the present moment, and with complementary probability $1- e^{-bdt} \approx bdt$ the investor will have to liquidate within dt. A similar random termination feature, although in discrete time setting, is used in Foucault, 1999. It allows one to analyze stationary equilibria of a dynamic order placement and execution model with potentially infinite number of trading rounds.

The investor objective at every moment t is to maximize the performance index

$$J(x_t^m, x_t^i) = \int_t^{\infty} e^{-bt} \{u(c_t) + bv(x_t^m, x_t^i)\} dt \ , \qquad (13)$$

subject to the appropriate state-transition equation and the initial cash holdings (x_t^m, x_t^i). Maximization is achieved by choosing the trajectory of dividends, active trades (purchases or sales of domestic against foreign currency through a market-maker) and, if the agent is a dealer, the trajectory of quotes.

Note that, thanks to the presence of liquidation function v in the current utility in (13), the problem does not require the imposition of transversality conditions: explosive x-paths are excluded by the defined properties of function v.

The state transition equations will be different depending on the investor category (dealer or not) and the market structure (decentralized or brokered). They are similar to the cash holding variable definitions of Section 2, but describe cash change rates instead of levels. Specifically, the non-dealer market user in the direct market has the state-transition equations

$$\dot{x}^m = y^m - f(r^1 + s^1 Q^1) Q^1 - f(r^2 + s^2 Q^2) Q^2 - c \,, \qquad (14a)$$

$$\dot{x}^i = y^i + Q^1 + Q^2 \,. \qquad (14b)$$

Dealer 1 in the direct market has state-transition equations (cf. (6))

$$\dot{x}^m = y^m + f\left(r^1 + g(r^1, q^2)\right)\frac{g(r^1, q^2)}{s^1} + f\left(r^1 + g(r^1, q^U)\right)\frac{g(r^1, q^U)}{s^1}$$
$$- f\left(r^2 + s^2 q^{12}\right) q^{12} + f(r^{c1}) h(q^{c1}) - c \,, \qquad (15a)$$

$$\dot{x}^i = y^i - \frac{g(r^1, q^2) + g(r^1, q^U)}{s^1} + q^{12} - q^{c1} \,. \qquad (15b)$$

The state-transition equations for the same dealer in the brokered market are (cf. (9))

$$\dot{x}^m = y^m + \frac{f\left(r^b + g(r^b, q^2)\right) g(r^b, q^1)}{s^1}$$
$$+ \frac{s^b}{s^1}\left\{ f\left(r^b + g(r^b, q^2)\right)\frac{g(r^b, q^2)}{s^b} + f\left(r^b + g(r^b, q^U)\right)\frac{g(r^b, q^U)}{s^b} \right\}$$
$$- f\left(r^b + s^b q^1\right) q^1 + f(r^b) q^{b1} + f(r^{c1}) h(q^{c1}) - c \,, \qquad (16a)$$

$$\dot{x}^i = y^i - \frac{s^b}{s^1} \frac{g(r^b, q^1) + g(r^b, q^2) + g(r^b, q^2)}{s^b} + q^1 - q^{b1} - q^{c1}. \quad (16b)$$

Note that (14a)-(16a) contain the dividend rate c, which was not defined in the one-period case. Another substantial difference compared to (2), (6) and (9) lies in the definition of marginal values q (with superscripts corresponding to individual market participants). The latter are defined as $q = \frac{x_i}{x_m}$, x being the adjoint variables of the corresponding optimization problem (see Section A2 of the Appendix for details). With this caveat in mind, the remaining state-transition equations can be defined by building an analogy with (2), (7), (9).

We proceed by using (13)-(16) and the analogous transition equations for the remaining market participants, to define a differential game between the non-dealer investor, dealer 1 and dealer 2 (actually, two games: one for the direct and the other for the brokered market). The payoff is given by (13) for both market organizations, every participant and every time moment. We shall consider open loop Nash equilibria of this game. This choice seems intuitively more appropriate for modeling FX dealer interaction than close loop equilibria, since the latter would imply that each dealer is able to evaluate the impact of his quoting and trading behavior on the actions of others. Such an ability would not be plausible in a multi-dealer environment with a limited degree of transparency. Moreover, we will concentrate specifically on steady state Nash equilibria in this class.

Proposition 3 *Let the differential inter-dealer game defined by payoffs* (13) *and state transition equations* (14)-(16) *(and their appropriate analogues), possess an open loop steady state equilibrium for, at least, a collection of finite intervals of constant cash endowment rates* y^{km}, y^{ki}, k=1, 2, *U. For each one-period inter-dealer game with initial cash endowments* $\tilde{y}^{km}$, $\tilde{y}^{ki}$, k=1, 2, *U, from a non-empty interval, there exists a set of parameters of the differential game such that the steady state Nash equilibrium of the latter corresponds to a Nash equilibrium of the former.*

This proposition and other properties of the steady state are proven in the Appendix, Section A3. The immediate technical value of the result consists in the possibility to simplify the Nash equilibrium search in the one-period game of Section 3. A direct calculation of the latter would involve a solution of a rather complex system of ten non-linear equations for ten unknowns (six cash holding variables and four quoting parameters). In the dynamic game, a part of the complexity is being removed since one characterizes Nash equilibria by means of three interrelated maximum principles (one for each market participant). Transition to the steady state Nash equilibrium means further simplification and dimensionality reduction.

There is one immediate application of Proposition 3. It exploits the said steady state Nash equilibrium feature of the one-period game in characterizing the so-called "hot potato" inter-dealer trades. In this setting, we define a hot potato trading pattern as an equilibrium outcome in which the gross inter-dealer order flow (the sum of net dealer orders) is bigger in absolute value than the net inter-dealer order flow. Formally, the gross inter-dealer flows in the direct and brokered market are equal, respectively, to $q^{12}+q^{21}$ and q^1+q^2, whereas the net inter-dealer flows are $q^{12}-q^{21}$ and q^1-q^2. The absence of hot potato trades would mean that at most one of the orders

q^{12}, q^{21} or q^1, q^2 is different from zero, i.e. a dealer would not place an order which is due to be offset by an opposite order in equilibrium. In the present model, however, Nash equilibria generically involve hot potato trades. This fact can be established in any equilibrium with given exogenous parameter values, when it is calculated numerically. The easiest, but also most spectacular outcome, is obtained analytically, when one considers the equilibrium with perfectly symmetric dealers. It turns out that hot potato trades are present even then.

Proposition 4 *Assume that the external matching (cf. 2.1.2 and 2.2.2) happens according to the same rule with the same parameters for both dealers, and that the quoting rule is exponential, as defined in 2.1. If the structural parameters and cash endowments of both dealers are identical, then the Nash equilibrium of the one period game is characterized by non-zero inter-dealer order flow* $q^{12}=q^{21}$ *in the direct market and* $q^1=q^2$ *in the brokered market every time the customer order flow* $Q^1=Q^2$ *or Q is itself non-zero.*

The proof, conducted directly for the steady state equilibrium of the dynamic game, is given in the Appendix, Section A4.

The result of Proposition 4 follows from the fact that, for a given dealer, making use of the market making function of the other dealer is always a dominating strategy with respect to abstaining from inter-dealer trade. Since the hot potato trades between symmetric dealers are Pareto-inferior to the no inter-dealer trade outcome, in a repeated one-period game there would exist a possibility of coordinating on a collusive no-trade outcome. However, this is hardly an intuitive outcome in a market with many competing dealers. In such a market, either the counter-party transparency is incomplete (brokered organization) or the chance of collusion is undermined by competition from other dealers (direct organization). Although, in this paper, we model just two dealers, it is done mainly for reasons of computational tractability (a discussion of Nash equilibrium properties obtained numerically follows in the next subsection). The overall objective is to study FX markets where one part of the participants may develop an intrinsic need to buy and the other to sell, which is expressed by different marginal utilities of foreign cash in equilibrium. So, our dealer 1 and dealer 2 just perform the roles of representatives for bigger groups with homogenous marginal currency values. In such an environment, coordination on a no-hot-potato trade equilibrium in a repeated game is as good as irrelevant.

To finish the comments on the differential game defined above, we note that its equilibrium trajectories are much more versatile than simple saddle paths converging to a steady state. In fact, one could identify three Lyapunov functions (corresponding to Hamiltonians of the three market participants) that must be constant along any open loop equilibrium trajectory. There probably are multidimensional attractors for the trajectories lying on the level surfaces of these three functions. The exact picture is so far unclear. In any case, almost all equilibrium trajectories of the dynamic game are periodic or quasi-periodic and do not have a single point of convergence. Therefore, a dynamic inter-dealer interaction in the set-up of this section exhibits enough complexity to match the observed volatility of the real forex.

To draw more specific qualitative conclusions, we now return to the one period set-up of Sections 2 and 3 and comment upon the findings obtained by solving for its Nash equilibrium numerically. Given the result of Proposition 3, the qualitative discussion to follow in the next subsection will utilize the outcomes of steady state calculations for the dynamic game, instead of the Nash equilibrium in the original one period game. We will discuss numerical results for dealer-symmetric NE in particular, since they are can be relatively easily illustrated graphically.

4.2 Equilibria of the one period game: the role of customer order flow under different market organizations

As was already mentioned, closed form solutions for Nash equilibria in the one period inter-dealer game do not exist. Numerically, one can derive equilibrium trading patterns by solving for the steady state Nash equilibrium in the "enclosing" differential inter-dealer game. We have done this for the interval of non-dealer foreign currency endowments that corresponds to her order flow (OF) values in the interval [-1,1].

Let us recall the pivotal role of the indirect marginal utility of foreign cash for the quoting and order placement behavior. To illustrate that role, we show in Fig. 2-5 how does the NE trading pattern depend on this marginal value of the non-dealer investor and, equivalently, on her order flow (in the dealer-symmetric equilibrium we are discussing now, there is a one-to-one correspondence between the two). These figures contain marginal FX values, order flows and transaction prices/exchange rates (ER) obtained by calculating Nash equilibria numerically for different exogenous characteristics of the non-dealer investors. The NE calculation results demonstrate the following similarities and differences of the two market structures, "in reduced form":

A. The effective transaction prices, the mid-quote and the marginal FX values of dealers and clients are all increasing functions of the client order flow, regardless of the trading mechanism. Accordingly, the dependence on the market user "demand intercept", represented by the marginal foreign cash value, is intuitive both in the direct and the brokered market.
B. Since the dealer parameters in the equilibrium we are discussing are identical, the inter-dealer trading is reduced to hot potato transactions. In the discussed equilibrium, they are represented by purchase orders that have the objective of compensating for the foreign currency holding reduction caused by customer purchases. This is true for both trading mechanisms.
C. Surprisingly, the hot potato orders in this equilibrium are still buying ones even when the client OF becomes negative (i.e. the non-dealer investors sell). This is so because client sales depress the price to levels that encourage dealers to buy.
D. Under any client order flow, inter-dealer OF in the brokered market is higher than in the direct market. That is, the same volume of client orders induces a higher inter-dealer activity in the brokered market. This means that this market is more "effort-consuming".
E. <u>Quoted dealer spreads</u> per unit of client order flow (parameters s of the model) are lower in the direct market when clients sell and in the brokered market when clients buy.
F. <u>Effective spreads</u>: Clients in the direct market pay a higher price margin over the mid-quote than dealers when they place a big buying order, i.e. *effective* client ask half-spreads in the direct market are higher than effective inter-dealer ask half-spreads, except for small orders. In the brokered market, the buy order volume for which the effective client ask half-spreads become higher than the inter-dealer half-spreads, is more elevated. (Effective bid half-spreads are hard to compare since, in this equilibrium, dealers only place buy orders.)
G. Altogether, the same level of client demand or supply corresponds to a smaller adjustment of the client marginal utility of foreign cash, which indicates that the brokered market has a

weaker ability of investor welfare improvement by means of market order placement, compared to the direct market.

We see that the two markets behave similarly in the qualitative sense but are characterized by different effective prices, spreads and inter-dealer trade volumes.

It is important to note that in the brokered market, the discussed Nash equilibrium is not the only one. For, at least, a certain interval of client OF values, there exist NE outcomes with exactly reverse dependence of the trade pattern on the client demand. In particular, a positive client order flow is accompanied by a price increase that generates *hot potato sales* instead of the buys in the previously discussed equilibrium. Generically, price and trade level in this equilibrium is depressed compared to both the direct market case and the "main" brokered market equilibrium case discussed previously.

Although we have not computed equilibria in a market with mixed direct-brokered organization (computational complexities forced us to relegate this calculation to subsequent research), one can conjecture that in such a market, dealers would buy in one and sell in another segment, still generating hot potato trades.

5. Conclusion

The paper investigated the relationship between FX transaction prices and order flows by means of a static and a dynamic model of inter-dealer quoting and trading game. The pivotal feature of the model was the presence of a global market user who traded with two competing dealers by splitting her orders optimally between them. We have studied both the direct inter-dealer and the brokered inter-dealer FX market cases.

The main result of the analysis is the key role of a dealer's current marginal valuation of the foreign cash (the ratio of marginal indirect utilities of the foreign and the domestic cash holdings) for the impact that the incoming order flow has on his price setting behavior. When dealers are symmetric, they still place orders with each other in equilibrium, generating the so-called "hot potato" trades. Asymmetry of marginal valuations, caused by either different endowments, information or the customer base composition, is the reason why there exists inter-dealer trade other than hot potato transactions. The marginal currency value is also the variable through which new orders, coming from both other dealers and the non-dealer customers, exercise impact on the quoted prices by channeling information about the fundamental changes in the currency supply and demand. Institutional differences between direct and brokered market structures are reflected in the quantitative relationships between the marginal values and the patterns of trade, but do not affect the main qualitative link between the marginal values, prices and equilibrium customer order flow.

Specifically, we have shown that

1. the price impact of the market user order flow is determined by the marginal indirect utility of foreign cash on the order placer and order recipient side.
2. Hot potato inter-dealer trading is a part of the equilibrium pattern of trade, i.e. it is present even when the dealers are completely identical. The reason is that the presence of at least one dealer makes it individually optimal for other dealers to unwind the FX position imbalances

through trade with him, rather than try to coordinate a no-trade outcome (a phenomenon of the prisoner dilemma type). In the Nash equilibrium with hot-potato trades, the latter force dealers to reveal their marginal currency valuations to other dealers.

3. The institutional arrangement of the forex makes a quantitative difference: for a given level of customer indirect utility value of foreign cash, the brokered market features higher inter-dealer order flow and lower customer prices. The direct market exhibits a stronger price response to customer orders than the brokered market. On the other hand, the qualitative dependence of order flow and price (exchange rate) on the marginal indirect utility level distribution is similar.
4. The above feature contrasts with the "reduced form" view of the dependence between the customer order-flow, inter-dealer trades and transaction prices. We find that in the brokered market, inter-dealer order flow is roughly twice as big as in the direct market, for a given level of client demand or supply. With the client order flow increasing, client effective prices surpass the inter-dealer effective prices earlier than in the brokered market. This comparative static assessment shows that the cost the investors have to bear for maintaining liquidity (and the price they have to pay for transparency) in the brokered market, exists in the form of discouragement of big market orders. Moreover, by placing the same market order, the non-dealer investor in the brokered market achieves a lower adjustment of her marginal FX utility. Consequently, brokered markets limit the participants' ability to improve welfare by trade, compared to direct markets.

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Appendix

A1 Proof of Propositions 1 and 2
Direct inter-dealer market

The problem of dealer 1 can be formulated as optimization of the objective function H^{d1} given by

$$H^{q1}(r,s) = \frac{L^{s1}(r;q^1,q^2,q^U)}{s} + f(r^{c1})h(q^{c1}) - q^1 q^{c1}$$

(see notation introduced at the beginning of Section 3), with respect to r, s, with parameters q defined in Subsection 2.1.1.

One can easily check the following auxiliary equalities:

$$\frac{\partial r^{c1}}{\partial r^1} = \frac{1}{2},\ \frac{\partial q^{c1}}{\partial r^1} = -\frac{1}{2s^1},\ \frac{\partial q^{c1}}{\partial s^1} = \frac{r^1 - g^1}{2(s^1)^2} = -\frac{q^{c1}}{s^1}.$$

Recalling the optimality properties of each market user's active trades, i.e. equations (4) for q=q^2 and q=q^U, we see that

$$\frac{\partial L^{s1}}{\partial r} = L_r^{s1} = L^{p1}.$$

Therefore,

$$\frac{\partial H^{q1}}{\partial r} = \frac{L^{p1}}{s} - \frac{1}{2s}\left[C^1(r,s) + D^1(r,s)\right],$$

and, for any fixed value of r, the above expression is equalized to zero for a single positive value of s (this partial derivative is positive for s below this critical value and negative for s above it). This proves (10a).

Similarly, it is immediately checked that

$$\frac{\partial H^{q1}}{\partial s} = \frac{1}{s^2}\left[\frac{r - g^1}{2}C^1(r,s) - L^{s1}(r;q^1,q^2,q^U)\right],$$

proving (10b). This completes the proof of Proposition 1.

Brokered inter-dealer market

The problem of dealer 1 is equivalent to maximizing

$$H^1(x^m,x^i,q^1,r^1,s^1,x_m,x_i) = v(x^m,x^i) + x_m\left[y^m + f(r^b)q^{b1} + f(r^{c1})h(q^{c1})\right]$$

$$+x_m\left[\frac{P(r^b,q^1)g(r^b,q^1) + P(r^b,q^2)g(r^b,q^2) + P(r^b,q^U)g(r^b,q^U)}{s^1} - f(r^b + s^b q^1)q^1\right],$$

$$+x_i\left[y^i - \frac{g(r^b,q^1) + g(r^b,q^2) + g(r^b,q^U)}{s^1} + q^1 - q^{b1} - q^{c1}\right]. \qquad \text{(A1)}$$

w.r.t. r^1 and s^1. Denote the right hand side of (A1) by $\mathbf{H}(r^b,r^1,s^1)$. The relevant partial derivatives of H^1 will be obtained with the help of the equality

$$\frac{\partial H^1}{\partial r^1} = \frac{\partial r^b}{\partial r^1}\frac{\partial \mathbf{H}}{\partial r^b} + \frac{\partial \mathbf{H}}{\partial r^1}. \qquad \text{(A2)}$$

It can be easily checked that

$$\frac{\partial r^b}{\partial r^1} = \frac{s^2}{s^1 + s^2}, \; \frac{\partial q^{b1}}{\partial r^1} = -\frac{1}{s^1+s^2}, \; \frac{\partial s^b}{\partial s^1} = \left(\frac{s^2}{s^1+s^2}\right)^2, \qquad \text{(A3)}$$

$$\frac{\partial r^b}{\partial s^1} = \frac{s^2(r^2 - r^1)}{(s^1+s^2)^2} = \frac{s^2}{s^1+s^2}q^{b1}, \; \frac{\partial q^{b1}}{\partial s^1} = \frac{r^1 - r^2}{(s^1+s^2)^2} = -\frac{q^{b1}}{s^1+s^2}. \qquad \text{(A4)}$$

Next, invoking (8) for q^1, q^2, q^U, we establish that

$$\frac{\partial \mathbf{H}}{\partial r^b} = x_m\left\{ f'(r^b)q^{b1} + \frac{1}{s^1}\left[\left(1+g_r(r^b,q^2)\right)q^2 + \left(1+g_r(r^b,q^U)\right)q^U\right]\right\}$$

$$-x_m\left\{\left(g_r(r^b,q^2) + g_r(r^b,q^U)\right)q^1 + P(r^b,q^2) + P(r^b,q^U)\right\} + x_m\frac{P(r^b,q^1) - q^1}{s^2}$$

$$= x_m\left\{ f'(r^b)q^{b1} + \frac{L^{p1}(r^b;q^1,q^2,q^U)}{s^1} + \frac{P(r^b,q^1)-q^1}{s^2}\right\},$$

$$\frac{\partial \mathbf{H}}{\partial r^1} = \frac{x_m}{s^1+s^2}\left(q^1 - P^b\right) - \frac{x_m}{2s^1}\left[C^1(r^1,s^1) + D^1(r^1,s^1)\right]. \qquad \text{(A5)}$$

Using the first two equalities in (A3), we get the following:

$$\frac{\partial H^1}{\partial r^1} = \frac{x_m s^2}{s^1(s^1+s^2)}\left\{L^{p1}(r^b;q^1,q^2,q^U) - B^1(r^1,r^2,s^1,s^2) - \frac{s^1+s^2}{2s^2}\left[C^1 + D^1\right]\right\},$$

proving (11a). To prove (11b) observe that

$$\frac{\partial H^1}{\partial s^1} = \frac{\partial r^b}{\partial s^1}\frac{\partial \mathbf{H}}{\partial r^b} + \frac{\partial \mathbf{H}}{\partial s^1} = \frac{\partial r^b}{\partial s^1}\left(\frac{\partial r^b}{\partial r^1}\right)^{-1}\left[\frac{\partial H^1}{\partial r^1} - \frac{\partial \mathbf{H}}{\partial r^1}\right] + \frac{\partial \mathbf{H}}{\partial s^1} \qquad \text{(A6)}$$

(the second equality follows from (A2)). Observe also that $\frac{\partial r^b}{\partial s^1}\left(\frac{\partial r^b}{\partial r^1}\right)^{-1} = \frac{r^2 - r^1}{s^1+s^2} = q^{b1}$.

Next, we get by direct calculation that

$$\frac{\partial \mathbf{H}}{\partial s^1} = x_m \left\{ f(r^b)\frac{\partial q^{b1}}{\partial s^1} - \frac{g(r^b,q^1)P(r^b,q^1)}{(s^1)^2} - f'(r^b + g(r^b,q^1))[g(r^b,q^1)]^2 \frac{1}{(s^b)^2}\frac{\partial s^b}{\partial s^1} \right\}$$

$$-x_m \frac{L^{s1}(r^b;q^1,q^2,q^U)}{(s^1)^2} + x_i \left\{ \frac{(r^1-g^1)C^1(r^1,s^1)}{2(s^1)^2} - \frac{\partial q^{b1}}{\partial s^1} \right\}.$$

Applying the last equality in (A3), the three equalities (A4) and the market order optimality condition $f'(r^b + g(r^b,q^1))g(r^b,q^1) = q^1 - P(r^b,q^1)$, the last expression can be reduced to

$$\frac{\partial \mathbf{H}}{\partial s^1} = x_m \left\{ \frac{(r^1 - r^2)(P^b - q^1)}{(s^1+s^2)^2} - \frac{L^{s1}(r^b;q^1,q^2,q^U)}{(s^1)^2} + \frac{(r^1-g^1)C^1(r^1,s^1)}{2(s^1)^2} \right\}.$$

However, by (A5),

$$\frac{\partial r^b}{\partial s^1}\left(\frac{\partial r^b}{\partial r^1}\right)^{-1}\frac{\partial \mathbf{H}}{\partial r^1} = x_m \frac{r^1-r^2}{s^1+s^2}\left\{\frac{C^1+D^1}{2s^1} + \frac{P^b - q^1}{s^1+s^2}\right\},$$

meaning that, by (A6),

$$\frac{\partial H^1}{\partial s^1} = \frac{x_m}{(s^1)^2}\left\{ \frac{(s^1)^2(r^1-r^2)(P^b-q^1)}{(s^1+s^2)^2} - L^{s1} + \frac{(r^1-g^1)C^1}{2} \right\}$$

$$+x_m \frac{r^1-r^2}{s^1+s^2}\left[\frac{q^1-P^b}{s^1+s^2} - \frac{C^1+D^1}{2s^1}\right] - x_m \frac{s^2}{s^1(s^1+s^2)}\left[L^{p1} - B^1 - \frac{(s^1+s^2)(C^1+D^1)}{2s^2}\right]$$

$$= \frac{x_m}{(s^1)^2}\left\{\frac{(r^1-g^1)C^1}{2} - L^{s1} - \frac{s^1 s^2 (r^1-r^2)}{(s^1+s^2)^2}\left(L^{p1} - B^1\right)\right\},$$

which gives us (11b) ?

A2 Trading strategies and equilibrium outcomes in the differential inter-dealer game

We shall use the Hamiltonian characterization of the solution to the optimal control problem (13) of a given investor, with state-transition equation (14), (15), (16) or analogous. For example, the current value Hamiltonian for dealer 1 in the direct market is defined as

$$H^U(x^m, x^i, c, Q^1, Q^2, x_m, x_i) = u(c) + bv(x^m, x^i)$$

$$+x_m\left[y^m - f(r^1 + s^1 Q^1)Q^1 - f(r^2 + s^2 Q^2)Q^2 - c\right] + x_i\left[y^i + Q^1 + Q^2\right], \quad \text{(A7)}$$

where x_m, x_i are adjoint variables of the problem. Their evolution is described by the Euler equations, to be featured shortly. We shall call these adjoint variables of the investor's optimization problem *shadow prices* of the domestic and foreign currency, respectively. They constitute the currency valuation by means of the investor's indirect utility.

Under the made assumptions about strict concavity of utility functions u and v and the growth properties of function v at infinity, the optimal policies of the global market user can be characterized by the first order conditions following from the Maximum Principle (see e.g. Fleming and Rishel, 1975):

$$u'(c) = x_m \,,\ \ f'\left(r^{\,j} + s^{\,j} Q^{j}\right)s^{\,j} Q^{j} + f\left(r^{\,j} + s^{\,j} Q^{j}\right) = \frac{x_i}{x_m}\,,\ j\text{=1,2.} \tag{A8}$$

The shadow prices are characterized by the adjoint equations

$$\dot{x}_m = b\left(x_m - \frac{\partial v}{\partial x^m}\right),\ \dot{x}_i = b\left(x_i - \frac{\partial v}{\partial x^i}\right) \tag{A9}$$

for every market participant, which is established by direct inspection of the maximum principle. The initial conditions are implicitly pinned down by the initial conditions for the state variables x^m and x^i.

Analogously, in the brokered market, the current value Hamiltonian optimization by dealer 1 implies the first order conditions of optimality

$$u'(c) = x_m \,,\ \ f'\left(r^{\,b} + s^{\,b} q^1\right)s^{\,b} q^1 + f\left(r^{\,b} + s^{\,b} q^1\right) = q^{\,1}$$

characterizing the dividend rate and own order size.

Propositions 1 and 2 of Section 3 are still valid, with the marginal currency values defined as

$$q^{\,k} = \frac{x_i^k}{x_m^k} = \frac{\partial V^k / \partial x^i}{\partial V^k / \partial x^m}\,,\ k\text{=1, 2, }U,$$

V^k being the value function of investor's problem (the result of maximization in (13)). These results are the two first order conditions on dealer 1's quoting parameters r^1 and s^1, and, by symmetry, two analogous conditions must be valid for dealer 2's quoting parameters r^2 and s^2.

The quoting parameters themselves depend on current and future values of q, i.e. to obtain the full characterization of equilibrium quotes, one needs to solve for the Nash equilibrium in the differential game. An embedded one-shot game at every time moment can be isolated if one fixes the marginal valuation vector $q=(q^1,q^2,q^U)$, since individual optimal orders and quoting parameters at each moment are well-defined functions of q. We are mostly interested in the embedded game corresponding to the steady state.

The steady state conditions for the Euler equations (A9) imply

$$\frac{\partial V^U}{\partial x^m} = x_m = v_m(x^m, x^i)\,,\ \frac{\partial V^U}{\partial x^i} = x_i = v_i(x^m, x^i)\,,\ q^{\,U} = \frac{v_m(x^m, x^i)}{v_i(x^m, x^i)} \tag{A10}$$

(as usual, subscripts denote partial derivatives). If we denote by j the inverse function to the marginal utility u' of the investor's dividend rate, then optimization of the latter in the steady state implies $c=j(v_m(x^m,x^i))$. These facts will be used in the next two sections.

A3 Proof of Proposition 3

Let $n=[r,s]^T$ be the 4-dimensional vector of optimal quoting rules, considered a function of the 3-dimensional vector q of the three marginal foreign currency valuations of the market participants: $n=N(q)$. Further, let $k(n)$ denote the terms in the cash position 6-equation system (2), (6) (in the direct market) or (7), (9) (in the brokered market), with the expressions (3) or (7) for the optimal active trades already substituted. That is, we write symbolically

$$\tilde{x}^m = \tilde{y}^m + k^m(n), \tag{A11a}$$

$$\tilde{x}^i = \tilde{y}^i + k^i(n), \tag{A11b}$$

where $\tilde{x}$ is the 6-dimensional vector of cash positions of the two dealers and the non-dealer market user, whereas $\tilde{y}$ is the 6-dimensional vector of their start-of-the-period cash endowments. For a given $\tilde{y}$, we are looking for such end of period cash holding vector $\tilde{x}$ that the Nash equilibrium defined at the end of Section 3, is attained. This means that the condition

$$\tilde{n} = N(q) = N \circ J(\tilde{x}) = M(\tilde{x}), \tag{A12}$$

must be satisfied for $J = \left(J^1, J^2, J^U\right)$, $J^l(x) = \dfrac{v_i(x)}{v_m(x)}$, l=1,2,U. Solution $\tilde{x}$ to the system (A11), (A12) is the Nash equilibrium of the one period inter-dealer game. To prove Proposition 4, we shall find an endowment rate vector y in the dynamic game such that its steady state Nash equilibrium quoting rule $\bar{n} = N(\bar{q})$ is equal to $\tilde{n}$. This is equivalent to requiring that the steady state NE cash holding vector $\bar{x}$ satisfies the equality $M(\bar{x}) = M(\tilde{x})$, equivalent to $J(\bar{x}) = J(\tilde{x})$.

First observe that (A11) can be written as $k(\tilde{n}) = \tilde{x} - \tilde{y}$, whereas the steady state NE in the differential game (first order conditions plus constancy of x and x) can be summarized as

$$k^m(\bar{n}) = j \circ v_m(\bar{x}) - y^m, \tag{A13a}$$

$$k^i(\bar{n}) = -y^i, \tag{A13b}$$

Note that (A13) is a non-trivial consequence of the maximum principle and the steady state conditions for variables x and x. It is at this point that the results of the dynamic game theory help us to comply with the NE equilibrium conditions in the one period game.

In order to obtain the one period game NE from the steady state NE, we must make sure that the right hand side of (A13) be equal to $k(\tilde{n}) = \tilde{x} - \tilde{y}$. This implies that the sought endowment rates y^m, y^i generate $\tilde{x}$ according to the rule

$$\tilde{x}^m = j \circ v_m(\bar{x}^m) + \tilde{y}^m - y^m, \ \tilde{x}^i = \tilde{y}^i - y^i, \tag{A14}$$

and, to prove our statement, we must find y that generate a steady state NE cash holdings $\bar{x}$ that satisfy $J(\bar{x}) = J(\tilde{x})$. Recalling (1) and once again invoking (A13), we restate the latter condition in the form

$$a_i k^i \circ N \circ J(\bar{x}) - a_m k^m \circ N \circ J(\bar{x}) = a_m \tilde{y}^m - a_i \tilde{y}^i. \tag{A15}$$

Now, observe that one can generate any vector $q = J(\bar{x})$ with strictly positive components by varying $\bar{x}$. So, if we find $q = J(\bar{x})$ to satisfy (A5), endowment rates y can be reconstructed from $\bar{x}$ using (A13). But, by checking that the map $q \mapsto a_i k^i \circ N(q) - a_m k^m \circ N(q)$ has a full rank Jacobian at least on the open, everywhere dense subset of $\mathbf{R}^{3+}$, we conclude that its range must be equal to the whole real line. Actually, there is a lot of freedom in the choice of $\bar{x}$, which means that one can generate different "convergence speeds" to the one period game NE. This concludes the proof ?

A4 Proof of Proposition 4

We will proceed by deriving the steady state equilibrium equation system separately for the direct and the brokered market mechanisms. Then we go over to the symmetric steady state case and prove the statement of Proposition 4 for the direct market. The proof for the brokered market is fully analogous.

Recall that the differential equations describing the optimal behavior of dealer 1, dealer 2 and the non-dealer market user are given by state-transition equations (14)-(16) (or their analogues) and (A9) (adjoint equations describing the evolution of the co-state variables for all three players).

The quoting rule f is exponential: $f(p) = e^{cp}$, c>0. Then the auxiliary function a introduced at the beginning of Section 3, reduces to

$$a(r,q) = 1 + g_r(r,q) = \frac{f'(r+g(r,q))}{2f'(r+g(r,q)) + g(r,q)f''(r+g(r,q))} = \frac{1}{2+cg(r,q)},$$

and the price function becomes

$$P(r,q) = f(r+g(r,q)) = e^{c(r+g(r,q))}.$$

We have assumed that the external matching happens according to the same rule with the same parameters for both dealers, so that the g-parameter is common for them. Let us rewrite the other two auxiliary functions, C and D, in terms of variables q^c and s instead of the original r and s. They become

$$C^1(r,s) = \left[1 - aq^{c1}\right]e^{c\frac{r+g}{2}} - q^1,\ D^1(r,s) = -csq^{c1}\left[1 - \frac{aq^{c1}}{2}\right]e^{c\frac{r+g}{2}}.$$

Similar notations (index 2 replacing 1) will be used for dealer 2.

We shall now go over to choosing a more convenient set of variables for which the steady state will be characterized.

First, looking at the structure of the steady state conditions for our set of differential equations, we note that (4a), (15a), (16a) relate the steady state values of the domestic cash shadow price, x_m, with the remaining variables, whereas this variable does not appear alone in any other equation. Therefore, one can replace the pair (x_m, x_i) by the pair (x_m, q) in further considerations, thus eliminating the necessity to involve the steady state versions of (14a)-(16a) in the calculations to follow.

Second, observe that for any agent, neither the right hand side of the state-transition equation for x^i, i.e. (14b), (15b), or (16b), nor the first order conditions of optimality, i.e. (3), (8), (10), or (11), depend on state variables x. Therefore, these equations can be used to pin down the three marginal values q and the two pairs of optimal quoting parameters, (r^k, s^k), k=1,2. Subsequently, the steady state conditions for the adjoint equations (A9) can be used to determine the steady state values of the state variables x. Accordingly, one does not need to include (A9) into intermediate calculations. One is left with variables $q^{c1}, s^1, q^{c2}, s^2, q^1, q^2, q^U$ and only deals with (14b)-(16b), (10) and (11).

Third, one shall observe that, most of the time, the marginal currency values q appear only as arguments in functions g. The only exceptions are functions L^p and L^s in (10), (11). In these expressions, it is more convenient to revert back from q to terms containing g only, by using the left hand side of the second equation in (A8). Instead of g, we shall use the trade volume variables q and Q directly, since it turns out to be simpler in terms of notation.

A4.2 Direct inter-dealer market

Rewriting (A8), we see that $q^1 = e^{c\left(r^2+s^2q^{12}\right)}(1+cs^2q^{12})$,

$$q^2 = e^{c(r^1+s^1q^{21})}(1+cs^1q^{21}),\ a(r^1,q^2)q^2 = (1-a(r^1,q^2))e^{c(r^1+s^1q^{21})},$$
$$q^U = e^{c(r^1+s^1Q^1)}(1+cs^1Q^1),\ a(r^1,q^U)q^U = (1-a(r^1,q^U))e^{c(r^1+s^1Q^1)},$$

and similar formulae can be written for dealer 2. This allows us to rewrite L^p and L^s as follows:

$$L^{p1} = \left(\frac{1+cs^1q^{21}}{2+cs^1q^{21}} + \frac{1+cs^1Q^1}{2+cs^1Q^1}\right)e^{c\left(r^2+s^2q^{12}\right)}\left(1+cs^2q^{12}\right) - \frac{e^{c\left(r^1+s^1q^{21}\right)}}{2+cs^1q^{21}} - \frac{e^{c\left(r^1+s^1Q^1\right)}}{2+cs^1Q^1},$$

$$L^{s1}(r,q) = s^1\left[e^{c\left(r^1+s^1q^{21}\right)}q^{21} + e^{c\left(r^1+s^1Q^1\right)}Q^1 - e^{c\left(r^2+s^2q^{12}\right)}\left(1+cs^2q^{12}\right)\left(q^{21}+Q^1\right)\right]$$

(and similarly for dealer 2).

The next step is substitution of the above expressions into the first order conditions (10), collecting terms that contain q^1, on the left hand side and dividing by $exp(cr^1)$. Use the fact that $r^k=g-2s^kq^{ck}$, $k=1,2$. The result is the following pair of optimality conditions:

$$\left(\frac{1}{2} + \frac{1+cs^1q^{21}}{2+cs^1q^{21}} + \frac{1+cs^1Q^1}{2+cs^1Q^1}\right)e^{2c\left(s^1q^{c1}-s^2q^{c2}\right)+cs^2q^{12}}\left(1+cs^2q^{12}\right)$$

$$= \frac{e^{cs^1q^{21}}}{2+cs^1q^{21}} + \frac{e^{cs^1Q^1}}{2+cs^1Q^1} + \frac{1-aq^{c1}-cs^1\left(1-\frac{aq^{c1}}{2}\right)q^{c1}}{2}e^{cs^1q^{c1}}, \qquad \text{(A16)}$$

$$e^{2c\left(s^1q^{c1}-s^2q^{c2}\right)+cs^2q^{12}}\left(1+cs^2q^{12}\right)\left(q^{21}+Q^1+q^{c1}\right) = e^{cs^1q^{21}}q^{21} + e^{cs^1Q^1}Q^1 + e^{cs^1q^{c1}}(1-aq^{c1})q^{c1}.$$

(A17)

In addition, the optimal trade volumes of the non-dealer market user are linked, in accordance with (7), by the equality

$$e^{c(g-2s^1q^{c1}+s^1Q^1)}(1+cs^1Q^1) = e^{c(g-2s^2q^{c2}+s^2Q^2)}(1+cs^2Q^2). \qquad \text{(A18)}$$

Note that (A16)-(A18) are valid outside the steady state as well, all we have done is a change of notations. To analyze the steady state, one must add the to (A16), (A17) and the corresponding pair of first order conditions for dealer 2 the three equations that state the constancy of the foreign cash holdings in the steady state. By denoting the foreign cash endowments of dealer 1, dealer 2 and the market user by y^{1i}, y^{2i} and Y^i for convenience, we write these conditions as

$$q^{21}+Q^1+q^{c1}=y^{1i}+q^{12}, \qquad \text{(A19)}$$

$$q^{12}+Q^2+q^{c2}=y^{2i}+q^{21}, \tag{A20}$$

$$Q^1+Q^2=-Y^i. \tag{A21}$$

A4.3 Brokered inter-dealer market

This time, (7) implies that

$$q^1 = e^{c\left(r^b+s^bq^1\right)}(1+cs^bq^1)\,,\; q^2 = e^{c(r^b+s^bq^2)}(1+cs^bq^2)\,,\; q^U = e^{c(r^b+s^bQ)}(1+cs^bQ)\,,$$

and

$$L^{p1} = \left(\frac{1+cs^bq^2}{2+cs^bq^2}+\frac{1+cs^bQ}{2+cs^bQ}\right)\left(1+cs^bq^1\right)e^{c\left(r^b+s^bq^1\right)} - \frac{e^{c\left(r^b+s^bq^2\right)}}{2+cs^bq^2} - \frac{e^{c\left(r^b+s^bQ\right)}}{2+cs^bQ},$$

$$L^{s1}(r,q) = s^b\left[e^{c\left(r^b+s^bq^2\right)}q^2 + e^{c\left(r^b+s^bQ\right)}Q - e^{c\left(r^b+s^bq^2\right)}\left(1+cs^bq^1\right)\left(q^2+Q\right)\right].$$

In the same way as in A4.2, we will use q^{c1}, q^{c2} instead of r^1, r^2 as unknown variables. It can be easily checked that $r^b=g-2s^b(q^{c1}+q^{c2})/2$ and the auxiliary function B^1 appearing in the first order conditions (11) can be expressed as

$$B^1 = e^{cr^b}\left\{\frac{s^1}{s^2}\left(1-e^{cs^bq^1}\right)+\frac{2cs^1\left(s^2q^{c2}-s^1q^{c1}\right)}{s^1+s^2}\right\}.$$

Therefore, after minor transformations, the first order conditions (11) can be written, analogously with (A16), (A17), as

$$\left(\frac{s^1+s^2}{2s^2}+\frac{1+cs^bq^2}{2+cs^bq^2}+\frac{1+cs^bQ}{2+cs^bQ}\right)e^{cs^bq^1}\left(1+cs^bq^1\right)+\frac{s^1}{s^2}\left(e^{cs^bq^1}-1\right)$$

$$=\frac{e^{cs^bq^2}}{2+cs^bq^2}+\frac{e^{cs^bQ}}{2+cs^bQ}+\frac{2cs^1\left(s^2q^{c2}-s^1q^{c1}\right)}{s^1+s^2}$$

$$+\frac{s^1+s^2}{2s^2}\left[1-aq^{c1}-cs^1\left(1-\frac{aq^{c1}}{2}\right)q^{c1}\right]e^{2cs^b\left(q^{c1}+q^{c2}\right)-cs^1q^{c1}}, \tag{A22}$$

$$e^{cs^bq^2}q^2+e^{cs^bQ}Q+e^{2cs^b\left(q^{c1}+q^{c2}\right)-cs^2q^1}(1-aq^{c1})\left(q^{c1}+q^{c2}\right)=e^{cs^1q^{c1}}\left(1+cs^bq^1\right)\left(q^2+Q+q^{c1}+q^{c2}\right).$$

$$+e^{2cs^b\left(q^{c1}+q^{c2}\right)-cs^1q^{c1}}cs^1\frac{s^2q^{c2}-s^1q^{c1}}{s^2}q^{c1}\left(1-\frac{aq^{c1}}{2}\right). \tag{A23}$$

Similar equations must hold for dealer 2.

These equations must be completed with the three steady state conditions following from (16b) and the (trivial) brokered-market analogue of (14b):

$$\frac{s^1}{s^1+s^2}q^1+\frac{s^2}{s^1+s^2}\left(q^2+Q\right)+\frac{(3s^1+s^2)q^{c1}-s^2q^{c2}}{s^1+s^2}=y^{1i}, \tag{A24}$$

$$\frac{s^2}{s^1+s^2}q^2+\frac{s^1}{s^1+s^2}\left(q^1+Q\right)+\frac{(3s^2+s^1)q^{c2}-s^1q^{c1}}{s^1+s^2}=y^{2i}, \tag{A25}$$

$$Q=-Y^i. \tag{A26}$$

In the direct market, (A16)-(A21) is a system of eight equations for eight unknown variables, namely q^{12}, q^{21}, Q^1, Q^2, q^{c1}, q^{c2}, s^1, s^2. In the brokered market, the system (A22)-(A25) has six equations for six unknowns: q^1, q^2, q^{c1}, q^{c2}, s^1, s^2. These systems must be solved to obtain the steady state Nash equilibria. In full generality this can only be done numerically.

A4.4 Dealer-symmetric steady state, direct market

Complete symmetry of both parameters and behavior of dealer 1 and dealer 2 implies that the market-user's trades with the two market makers are also identical. Accordingly, we have only four unknown variables instead of the original eight: q – the inter-dealer trade volume, Q – the customer trade volume, q^c – the volume of trade in external matching (looking for q^c is equivalent to looking for r, the dealer mid-quote), s - the slope of the dealer pricing schedule. Since (A18) is now vacuous and (A19), (A20) become one, we are also left with only four equations: (A16), (A17) (both simplified), (A19) and (A21). Denoting by y^i the now common per-period dealer endowment with foreign cash, we observe that Q and q^c are fully determined by the steady state conditions (A19), (A21): $q^c=y^i+\frac{Y^i}{2}$, $Q=-\frac{Y^i}{2}$. The corresponding simplified versions of (A16) and (A17) look like

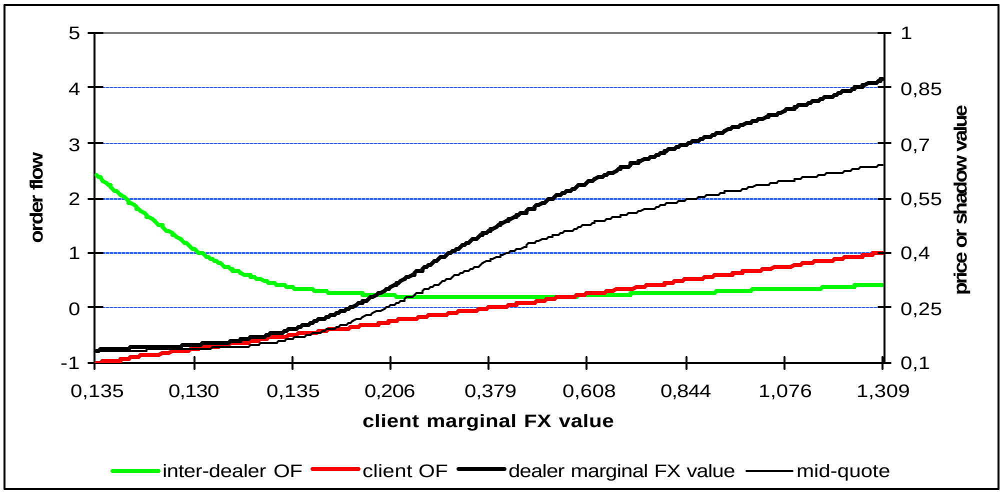
$$\left\{\left(\frac{1}{2}+\frac{1+csq}{2+csq}+\frac{1+csQ}{2+csQ}\right)(1+csq)-\frac{1}{2+csq}\right\}e^{csq}=\frac{e^{csQ}}{2+csQ}+\frac{1-aq^c-cs\left(1-\frac{aq^c}{2}\right)q^c}{2}e^{csq^c} \tag{A27}$$

$$\left[(1+csq)(q+y^i)-q\right]e^{csq}=e^{csQ}Q+e^{csq^c}(1-aq^c)q^c. \tag{A28}$$

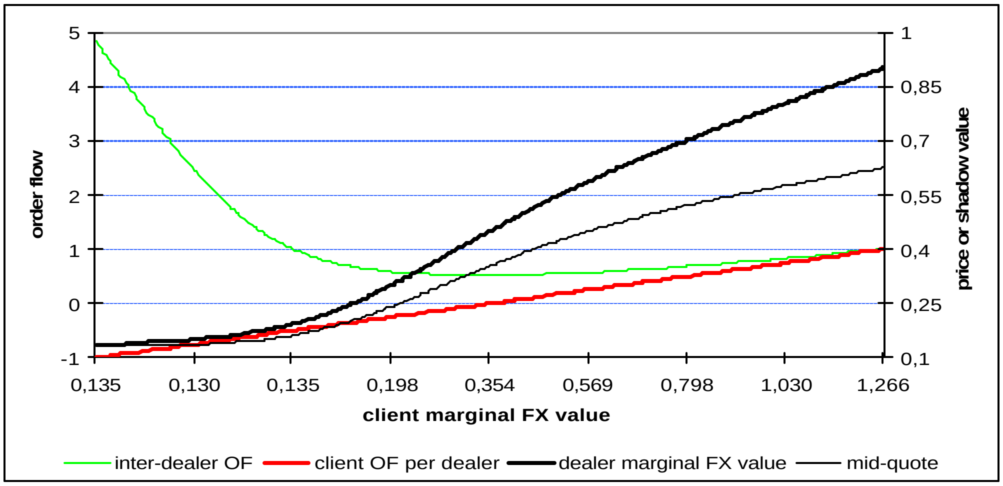
This is a system of two equations for two unknowns, q and s. Except for an exceptional combination of exogenous parameters, including the condition Y^i=0 that was excluded from consideration in Proposition 4, q=0 cannot be a part of the solution for the system (A27), (A28). That is, generically, "hot potato" trade cannot be avoided ?

Fig. 2 Trade patterns as a function of the investor indirect marginal utility of foreign currency holdings

(a) Direct inter-dealer market



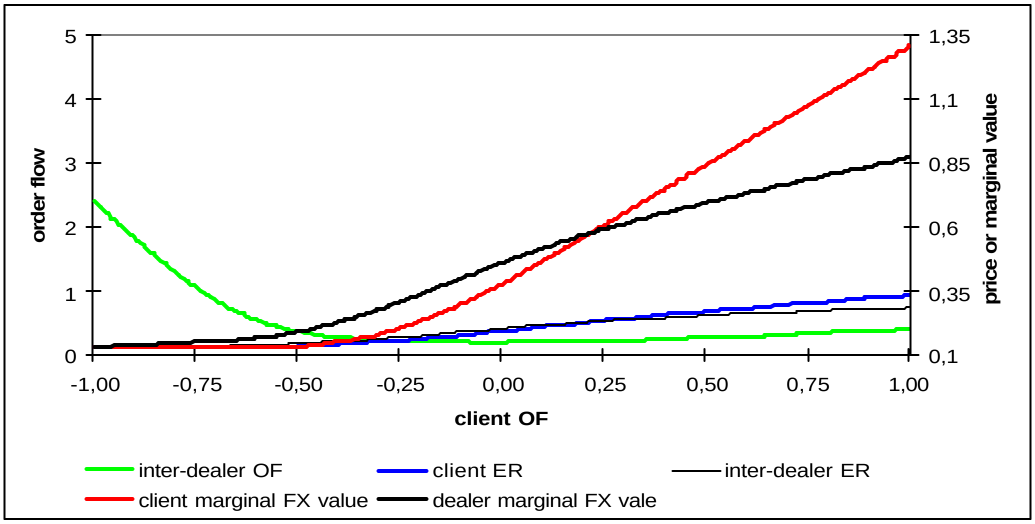
(b) Brokered inter-dealer market



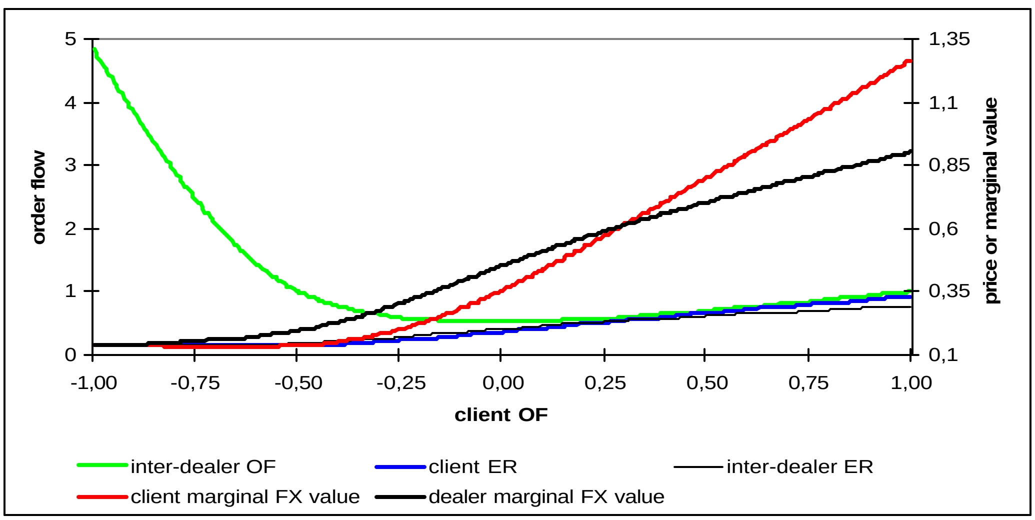
Note: The figure shows the trade pattern outcomes of the one period quoting and trading game, corresponding to different values of the marginal foreign cash valuation of the representative non-dealer investor. The marginal foreign currency value is the endogenous indirect utility of the non-dealing investor obtained in the Nash equilibrium, and is in a one-to-one correspondence with this investor's per-dealer order flow, which is the negative of the exogenous investor foreign cash endowment.

Fig. 3 Trade patterns as a function of the non-dealer investor order flow per dealer

(a) Direct inter-dealer market



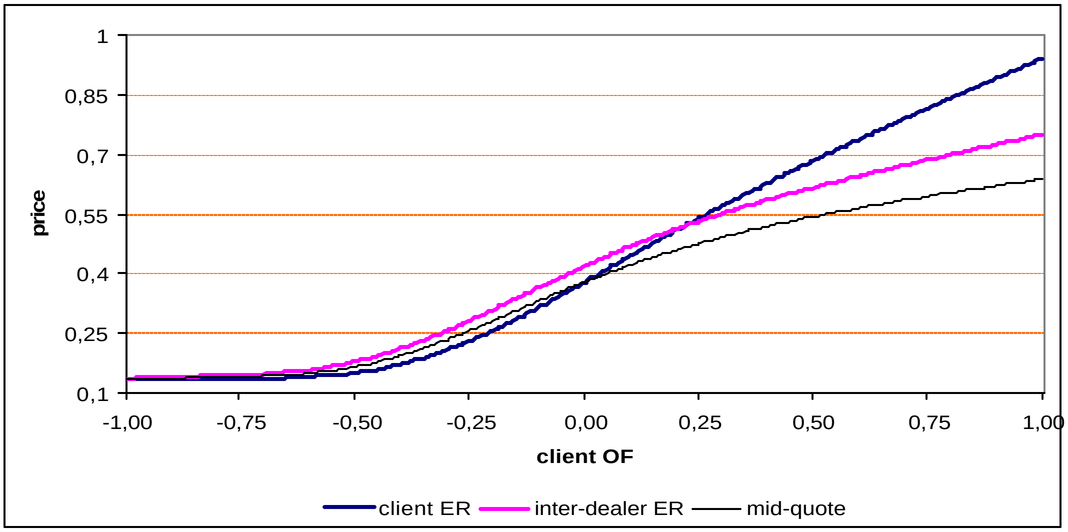
(b) Brokered inter-dealer market



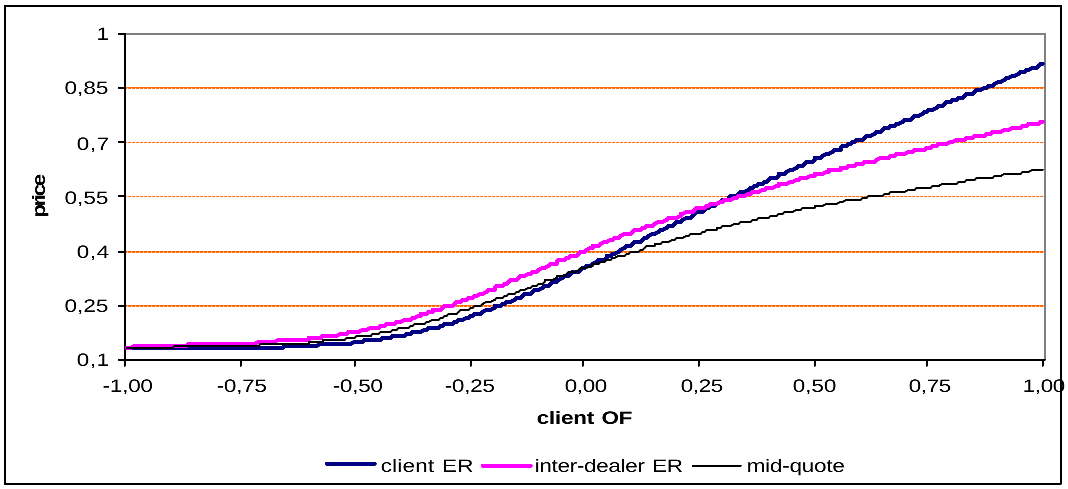
Note: The figure shows the trade pattern outcomes of the one period quoting and trading game, corresponding to different values of the representative non-dealer investor per-dealer order flow, which is the negative of the exogenous investor foreign cash endowment.

Fig. 4 Exchange rate setting and effective transaction prices as functions of the non-dealer investor order flow per dealer

(a) Direct inter-dealer market



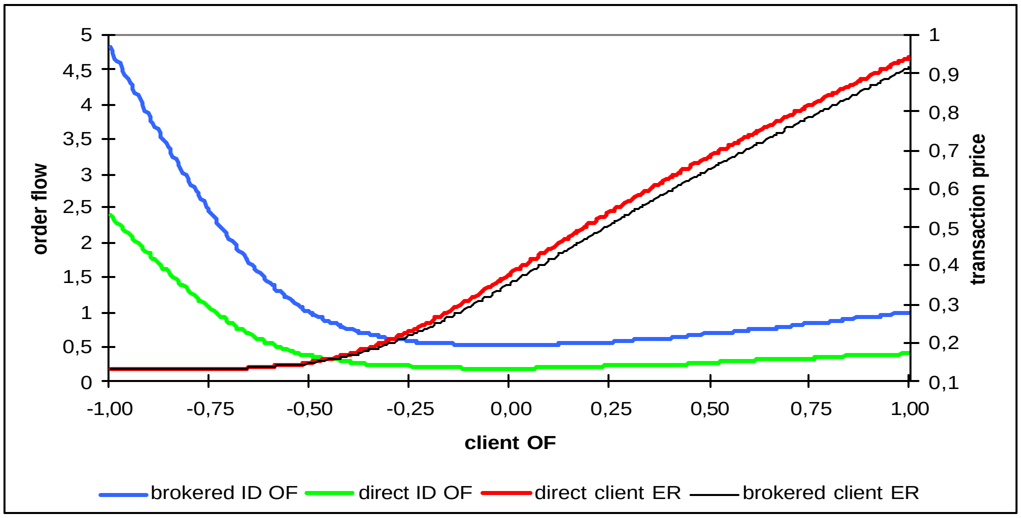
(b) Brokered inter-dealer market



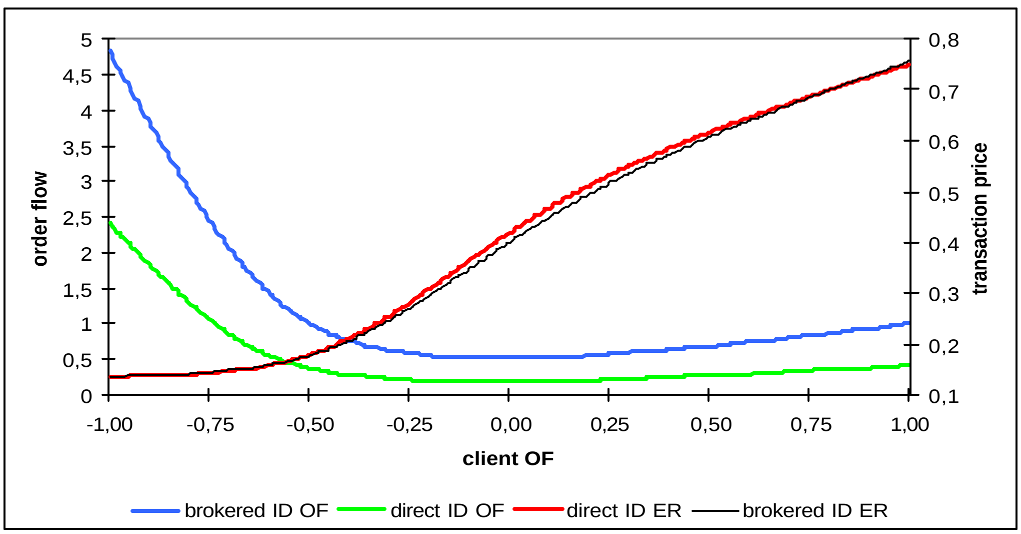
Note: The figure shows the effective customer-dealer and inter-dealer transaction prices and the exchange rate mid-quote as functions of the customer order flow.

Fig. 5 Comparison of the direct and brokered trading mechanisms for a given level of client's order flow per dealer

(a) Inter-dealer order flows and effective exchange rates paid by the client



(b) Inter-dealer order flows and effective exchange rates paid by the dealer



Note: The figure shows the dealer order flows and effective customer-dealer (first panel) and inter-dealer (second panel) transaction prices as functions of the customer order flow.